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Standard covariant formulation for perfect-fluid dynamics

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After a brief description of the Milne generalization of the Galilean invariance group for the space–time of Newtonian kinematics, it is shown how the generalized Eulerian dynamical equations for the motion of a multiconstituent perfect (non-conducting) fluid can be expressed in terms of interior products of current 4-vectors with exterior derivatives of the appropriate 4-momentum 1-forms (whose role is central in this approach) in a fully covariant standard form whose structure is identical in the Newtonian case to that of the corresponding equation for the case of (special or general) relativistic perfect fluid mechanics. In addition to space–time covariance, this standard form exhibits chemical covariance in the sense that it is manifestly invariant under redefinition of the number densities of the independent conserved chemical constituents in terms of linear combinations of each other. It is shown how, in the strictly conservative case when no chemical reactions occur, this standard form, can be used (via the construction of suitably generalized Clebsch potentials) for setting up an Eulerian (fixed-point) variation principle in a form that is simultaneously valid for both Newtonian and relativistic cases.

1. Introduction

The purpose of this work is to show how the equations of motion of a non-conducting perfect fluid with an arbitrary number of independent (possibly charged) conserved constituents may be expressed in a very convenient standard form characterized by the following useful properties:

(i) It manifests chemical covariance, meaning that it is unaffected by linear transformations of the constituent basis (in the sense described at the end of §2).

(ii) It is also geometrically covariant, meaning that it is covariant under general (not necessarily linear) transformations of the space–time coordinates (being expressed entirely in terms of interior and exterior products and derivatives in the sense of Cartan).

(iii) It is applicable not only to (special and general) relativistic fluid mechanics (as one would expect, in view of the preceding property) but also (less obviously) to ordinary Newtonian fluid mechanics.

As a consequence of these properties, and particularly of the last one, this standard formulation can be used as a starting point for the derivation of a large number of general theorems in such a way that they apply automatically both to the Newtonian

and the relativistic cases. The present work has in fact been essentially motivated by the observation (by several workers, and most recently by Katz 1984 and Bekenstein 1986: see also Bekenstein & Oron 1977) that the technical effort involved in derivation of the many known kinds of conservation law (see e.g. Schutz 1980; Katz & Lynden-Bell 1982; Gaffet 1985) for sufficiently well-behaved (non-dissipative) fluid models within the context of Newtonian theory (including laws of conservation of 'circulation' (Kelvin 1910), 'vorticity potential' (Ertel 1942), and 'helicity' (Moffatt 1969)), is actually greater – i.e. their derivations are technically more complicated – than for their fully relativistic analogues (as given for the quoted examples by Lichnerowicz 1967, Friedman 1978, and Carter 1979, respectively).

The existence of this shared standard formulation makes it possible to simplify the derivation of many known results in Newtonian hydrodynamics – and should facilitate the discovery of new ones – by enabling one to carry over the convenient methods previously developed (using a covariant treatment) in a relativistic context. As an illustration we shall first show explicitly in §§6 and 7 how the standard formulation leads directly to a corresponding very generally applicable Clebsch-type formulation and hence to a related Eulerian variation principle using the volume integral of the fluid pressure P. Finally, in §8, it will be shown how, when there are no more than two independent constituents, the standard form of the hydrodynamic equations may be converted into the particularly convenient (albeit no longer chemically covariant) canonical form as previously derived by Carter (1979) on the basis of the property that the individual fluid flow trajectories satisfy an ordinary Lagrangian variation principle in five dimensions, the extra dimension representing a canonical, not necessarily proper, time parametrization which must be introduced independently of the ordinary coordinate time in order to preserve general covariance.

The apparent paradox of the comparative simplicity of the physically more general case of the relativistic theory as compared with the more specialized case of the traditional Newtonian theory applies not only to fluids but also more generally to solid media (see e.g. Carter 1980) and can be largely understood as resulting from the fact that the latter is an essentially singular limit of the former: the existence of a non-degenerate 4-metric makes it easier to provide a simple fully covariant treatment in the relativistic case. However a significant further simplification is obtainable for non-dissipative fluid models (as for the more widely known example of Maxwellian electromagnetism) from the fact that the system can be formulated in a way that is naturally adapted to treatment by the Cartan method based on the exterior calculus of differential forms: like the more specialized canonical formulation (whose systematic exploitation was described by Carter 1979) the appropriately adapted standard formulation to be presented here has the advantage of involving only exterior differentiation, requiring no explicit reference to covariant differentiation in the Riemannian (i.e. connection dependent) sense, nor to the 4-dimension pseudo-Riemannian metric tensor which of course ceases to exist in the Newtonian limit. It was this consideration that led us to hope that such a formulation would have the well-behaved Newtonian limit that we have successfully obtained in the manner described in §4. The main part of the present text is in fact presented in such a way that (although the mathematical inspiration is relativistic) no physical postulates beyond those of Newtonian theory are invoked explicitly. For readers interested in the (essentially more straightforward) case of (special or general) relativistic hydrodynamics in its own right, the relevant derivation of the standard formulation is given in Appendix A.

2. Characterization of the standard formulation

In any (Newtonian or relativistic) description of a non-conducting fluid, the basic independent variables may be taken to consist on the one hand of the components u^ρ say ($\rho = 0, 1, 2, 3$) of the space–time tangent vector to the flow world lines, as normalized with respect to proper time, and on the other hand the proper number densities, n_X say, where X is an index labelling the various independent constituents involved (which may include quantities such as entropy as well as material particles), so that the current vectors representing the corresponding constituent fluxes will be given by

$$n^\rho_X = n_X u^\rho. \tag{2.1}$$

The creation, destruction or conservation of any such constituent will be governed by a four-dimensional divergence equation of the form

$$\nabla_\rho n^\rho_X = r_X, \tag{2.2}$$

where we use the symbol ∇ to denote covariant differentiation (with respect to the ordinary flat space–time connection in Newtonian theory or the usual Riemannian connection in relativistic theory) and where r_X is its numerical creation rate per unit volume. In order for the fluid to be perfect in the strict sense of having a reversible evolution, as will be required for the derivation of the potential formulation in §6, the variation principle in §7, and the canonical formulation in the case discussed in §8, the relevant constituents must all be conserved, i.e. the corresponding rates r_X must all vanish. On the other hand everything given in the other sections, and in particular all the basic formulae quoted below in this introductory section, will be valid even in the presence of non-zero creation rates arising e.g. from chemical or (in the astrophysical applications with which the present authors are most familiar) from nuclear reactions.

An essential ingredient of the present treatment will be the introduction for each constituent, as labelled by X, of a corresponding space–time convector field (a 1-form in the terminology of Cartan) with components π^X_ρ say which will be interpretable as representing the 4-momentum per particle for the constituent X in question. This 4-momentum is dynamically conjugate to the corresponding current 4-vector n^ρ_X with respect to the pressure P of the fluid in the same way that the electromagnetic 4-potential A_ρ is conjugate to the electric current 4-vector j^ρ: at a given space–time position in a fixed gravitational background the infinitesimal variation in the pressure determined by the equation of state as a result of corresponding variations in the 4-momenta and electromagnetic 4-potential is expressible as

$$\delta P = -n_X u^\rho \delta \pi^X_\rho + j^\rho \delta A_\rho, \tag{2.3}$$

where it is to be understood that the convention of summation over all values of an index that is repeated (in upper and lower positions) applies not only to the (lower-case Greek) space–time coordinate indices but also to the (capital Latin) chemical constituent indices, and where the total current 4-vector j^ρ is itself given in terms of a set of constants e^X representing the electric charge (if any) per unit particle number for each corresponding constituent X by the expression

$$j^\rho = e^X n^\rho_X = \rho_e u^X, \quad \rho_e = e^X n_X, \tag{2.4}$$

In terms of the quantities thus introduced, what we refer to as the standard form of the equations of motion is given by the expression

$$n_X u^\rho (\nabla_\rho \pi^X_\sigma - \nabla_\sigma \pi^X_\rho) = 0, \tag{2.5}$$

in which the covariant differentiation operations, that we have indicated by the symbol ∇, are interchangeable with simple partial differentiation operations, which we shall indicate by the symbol ∂, as a result of the antisymmetrization within the brackets, whose contents effectively represent the exterior derivatives of the 4-momenta. A closely related alternative version of the equations of motion (2.5) expresses the evolution in terms of the concept of Lie differentiation with respect to the flow tangent 4-vector u^ρ, which we shall indicate by the symbol $[\boldsymbol{u}\mathscr{L}]$. This alternative expression takes the form

$$n_X([\boldsymbol{u}\mathscr{L}]\pi^X_\rho - \partial_\rho \lambda^X) = 0, \tag{2.6}$$

where we have introduced a set of scalar fields

$$\lambda^X = u^\rho \pi^X_\rho, \tag{2.7}$$

that will be seen to be interpretable in a certain sense as representing individual particle Lagrangian functions for the corresponding constituents.

The crucial step in the present work is the reduction of the usual Newtonian mechanical equations of motion of the fluid to the standard form (2.5). This is carried out in §4, after a brief introductory discussion of the notion of the Milne extension of the (more widely known but unnecessarily restricted) Galilean invariance group in §3. By comparison with the Newtonian case (which is complicated by the absence of a non-degenerate space–time metric) the reduction to standard form of the equations of motion of a relativistic fluid is technically a much more straightforward exercise, which we shall postpone to the Appendix in order not to interrupt the continuity of the presentation for any readers who may be interested only in the Newtonian case. (In so far as the relativistic case is concerned, the form (2.5) is an obvious generalization of the expressions whose utility was emphasized some time ago by Lichnerowicz (1967). The fact that subsequent authors [e.g. Misner, Thorne & Wheeler 1973] have generally tended to ignore this approach is perhaps due in part to an unfortunate choice of nomenclature by Lichnerowicz, who used the physically misleading term 'current', and the corresponding symbol $\mathscr{C}$, to denote what we have referred to as the 4-momentum and denoted by the symbol π. In usual physical terminology the term current is appropriately applicable only to a contravariant vector field, or at a more fundamental level to the corresponding 3-form to which it is related via the space–time 4-volume measure. In so much as (2.3) shows that the 1-forms representing the 4-momenta are dynamically conjugate to genuine physical currents, it would evidently be more suitable to describe the momenta as co-currents.)

Provided no net charge is carried off from the fluid in the course of any reactions that may occur, i.e.

$$\nabla_\sigma j^\sigma = 0, \quad e^X r_X = 0, \tag{2.8}$$

then it will follow from the results given below that there will be an analogous conservation law for the net 4-momentum, i.e. we shall have

$$\pi^X_\rho r_X = 0. \tag{2.9}$$

The present approach is particularly effective for the purpose of obtaining rapid and obvious derivations (such as those demonstrated by Carter 1979) of results of

very general validity whose more widely known special cases are commonly obtained by inelegant or devious *ad hoc* methods. As a simple example, we present what may be described as a very general extension of the classical Bernoulli theorem, which will apply whenever the flow is invariant with respect to transport by some space–time 4-vector with components k^ρ say, so that in particular the corresponding Lie derivative of the 4-momentum 1-form will be zero, i.e.

$$[\boldsymbol{k}\mathscr{L}]\pi_\rho^X = 0, \tag{2.10}$$

which (by the same Cartan formula that was used to go from (2.5) to (2.6)) is equivalent to the condition

$$(\nabla_\rho \pi_\sigma^X - \nabla_\sigma \pi_\rho^X)\, k^\sigma = \nabla_\rho(k^\sigma \pi_\sigma^X), \tag{2.11}$$

where the scalars $k^\sigma \pi_\sigma^X$ appearing on the right-hand side would be interpretable for each corresponding constituent as for example the effective energy per particle if k^ρ were a generator of time transportation, or as angular momentum if it were an axial rotation generator. Under such conditions it can be seen immediately that the result of contracting (2.5) with k^ρ can be rewritten as

$$n_X u^\rho \nabla_\rho(k^\sigma \pi_\sigma^X) = 0. \tag{2.12}$$

Provided (2.8) is satisfied we may use (2.9) with (2.2) to convert this into a simple flux conservation law (for energy or whatever it may be) of the form

$$\nabla_\rho(k^\sigma \Pi_\sigma u^\rho) = 0, \tag{2.13}$$

where we have introduced the total 4-momentum density 1-form as defined by

$$\Pi_\rho = n_X \pi_\rho^X. \tag{2.14}$$

The chemical covariance referred to at the beginning concerns redefinitions of the elementary constituents of the fluid in terms of fixed linear combinations of their number densities by transformations in which the set of number densities n_X are replaced by a new set n'_X given by

$$n'_X = N_X^Y n_Y, \tag{2.15}$$

where the numerical coefficients N_X^Y are constants, which automatically ensures that the corresponding new currents, n'^ρ_X, will retain the same conservation properties as the original ones, or in other words, that (2.2) is covariant under the transformation. The construction of the 4-momenta will be such that they transform contravariantly under the effect of (2.6), i.e. the π_ρ^X will be replaced by a corresponding new set $\pi_\rho'^X$ given by

$$\pi_\rho^X = \pi_\rho'^Y N_Y^X. \tag{2.16}$$

We shall adopt the usual convention (analogous to that which is already implicitly understood to apply to the familiar lower-case Greek space–time coordinate indices) of using the upper or lower positions of the capital Latin indices to indicate respectively covariant or contravariant transformation properties under the effect of (2.15). Since the charges e^X are obviously contravariant in this sense, the chemical invariance of the total electric current, as expressed by (2.4) is manifest. In view of the constancy of the transformation coefficients, the chemical invariance of (2.3) and of the forms (2.5) and (2.6) is also apparent. (As a very simple example of the kind of constituent basis transformation to which one might wish to apply these considerations, suppose one has in mind a problem involving the transport of oxygen dissolved in water: as a first choice one might use a molecular basis for the

accounting, taking n_1 to be the number density of water in units of H_2O and taking n_2 to be the number density of oxygen in units of O_2; if as a second choice one decided to change to book-keeping on an atomic basis, taking n'_1 to be the number density of hydrogen atoms and n'_2 to be the number density of oxygen atoms, then the corresponding chemical transformation matrix would be given by $N^1_1 = N^2_2 = 2$, $N^1_2 = 1$, $N^2_1 = 0$.)

3. Newtonian kinematics and Milne transformations

The space–time background of Newtonian mechanics is traditionally described in terms of a linear structure subject to the (ten-parameter) Galilean transformation group, but a more modern approach whose development is due principally to the work of Milne, McRea and Bonnor (Milne 1934; McCrea & Milne 1934; Bonnor 1957) is based on recognition of a much larger (infinite-dimensional) invariance group of nonlinear transformations whose significance does not seem to have been noticed before Einstein's formulation of the 'equivalence principle' in the early years of the present century. In addition to the linear transformations of the Galilean group, this larger class of transformations, which we shall refer to as the Milne equivalence group, includes (as an infinite-dimensional invariant subgroup) the set of arbitrarily (i.e. in general nonlinearly) time-dependent space translations. The Milne group is thus intermediate between the Galilean group (which it contains as a non-invariant subgroup) and the full Einstein equivalence group of all diffeomorphisms of space–time (within which the Milne group is itself contained as a non-invariant subgroup).

The space–time background of the Einstein, Milne, McRea, Bonnor version of Newtonian mechanics can be conveniently described within the conceptual framework of fibre-bundle theory (which has been made familiar to physicists by modern gauge field theories). The salient feature of Newtonian (as opposed to relativistic) space–time, from both the traditional (Galilean) and the modern (Milne) point of view, is the existence of a preferred (proper) time coordinate field, t say (defined modulo an arbitrary additive constant) which specifies a foliation by (three-dimensional) constant-time slices each of which is endowed with a flat Euclidean geometric structure. The distinction between the traditional and the modern version concerns the relationship between neighbouring slices. The modern version can be succinctly described by the statement that the space–time as a whole has the structure of a fibre-bundle over a one-dimensional base manifold, parametrized by the time, t, with the Euclidean constant-time slices as fibres, subject to the gauge action of the (3-parameter) group of Euclidean translations. Provided one also requires that the one-dimensional time-base manifold be complete, it is evident that the structure set up in this way will be unique and furthermore that it will be invariant under the global action of the direct product of the 1-parameter group of time translations with the full 6-parameter Euclidean group (of which the locally acting 3-parameter gauge group is an invariant subgroup).

For any given choice of gauge (i.e. of local direct product structure) in the bundle that has just been described, a choice of space coordinates, x^i say ($i = 1, 2, 3$), in one of the fibres (i.e. at some instant) will determine corresponding coordinate values throughout the gauge patch (i.e. over the whole range of time under consideration). If the coordinates were originally chosen to be linear (with respect to the flat

Euclidean space-metric structure) then the effect of a bundle gauge transformation will be expressible imply by

$$x^i \to x^i + z^i, \tag{3.1}$$

where the z^i are arbitrary differentiable functions depending only on the base coordinate, t.

In the traditional Galilean approach it is supposed that there exists a preferred second-order connection, i.e. a connection not on the primary space–time bundle itself but on the associated bundle of section gradients, i.e. 3-velocities (since a bundle section is interpretable as the trajectory of a single particle in space as a function of time) with components given by

$$v^i = \mathrm{d}_0 x^i, \quad \mathrm{d}_0 \equiv \mathrm{d}/\mathrm{d}t. \tag{3.2}$$

(The existence of a preferred second-order connection is traditionally expressed by the statement that 'velocity is relative but acceleration is absolute'.) Since the time-base manifold is one-dimensional, such a connection is automatically integrable and thus determines a preferred direct product structure on the associated velocity bundle. This in turn determines a corresponding family of preferred (inertial) gauges for the primary space–time bundle. This set of preferred space–time bundle gauges, and the underlying (inertial) connection on the associated velocity bundle will be preserved by a finite (Galilean) subgroup distinguished within the infinite group of all local gauge transformations by the restriction that the time dependence of the z^i in (3.1) be linear.

In the Milne, McRea, Bonnor approach it is recognized that in the presence of gravitational fields (and in the absence of cosmologically implausible asymptotic boundary conditions) ordinary Newtonian mechanical theory does not actually contain any physical prescription (in any case not at a local level) for recognizing the preferred inertial gauges whose existence is traditionally postulated. (When the theory is extended to include Clerk Maxwell's electromagnetic field equations the physics does indeed specify a locally well-defined connection, but this Maxwellian (ether) connection is of ordinary first-order type, and as such not only breaks the full space–time bundle invariance, but even destroys the Galilean invariance.) This does not mean that it is impossible to define the absolute acceleration, meaning a Milne-gauge covariant velocity derivative, because the gravitational field, with components g^i say, is postulated to vary according to the rule

$$g^i \to g^i + (\mathrm{d}_0)^2 z^i, \tag{3.3}$$

under the effect of the Milne gauge transformation law (3.1), thereby making it possible to construct a corresponding covariant derivative:

$$\mathrm{D}_0 v^i \to \mathrm{D}_0 v^i, \quad \mathrm{D}_0 v^i \equiv \mathrm{d}_0 v^i - g^i. \tag{3.4}$$

In the trivial case when it is spacially uniform (i.e. independent of the Cartesian space coordinates x^i) a field g^i satisfying (3.3) will be interpretable as the gauge form of an effectively preferred connection which will then indeed specify a corresponding subset of inertial gauges within the general class of Milne gauges. However in general, when a non-trivial, i.e. spatially variable gravitational field is present, there is no (local) physical prescription for decomposing g^i into a spatially uniform part, interpretable as an inertial connection, and a spatially variable residual part, interpretable as the truly gravitational field.

Apart from its gauge transformation behaviour, our only other requirement on the

Newtonian gravitational field will be that the corresponding spatially covariant field with components given by g_i should be spatially irrotational, which means that it is derivable from a scalar potential,

$$g_i = -\nabla_i \Phi, \tag{3.5}$$

where index raising and lowering and covariant differentiation are defined with respect to the flat Euclidean structure on the fibres. (If the linear coordinates, to which the validity of (3.1) and (3.3) is limited, are actually taken to be Cartesian, then g_i and g^i may be identified.) The transformation law (3.3) implies a corresponding law of the form

$$\Phi \to \Phi - x^i (\mathrm{d}_0)^2 z_i, \tag{3.6}$$

for the effect of the Milne-gauge transformation (3.1) on the Newtonian potential ϕ.

In the present work we shall be concerned only with the effect of background gravitational and electromagnetic fields on the material (fluid) system under consideration, so that the source equations for the active effect of the matter on these fields will not play any role. However, before leaving the topic of the Milne-gauge dependence of the gravitational field we mention the obvious invariance of second space derivatives of Φ, and in particular of the Laplacian that the Poissonian source equation sets proportional to the material mass density which of course is also unaffected by Milne transformations. In so far as electromagnetic effects are concerned the status of the source equations is less satisfactory, because the incorporation of the Maxwellian source equations into a Newtonian background entails the violation not only of Milne covariance but even of Galilean covariance (whence the original theoretical motivation for developing relativistic theory). Nevertheless for the present purpose, in which the source equations play no role, there is no obstacle to including the effects of an electromagnetic background field in a manner entirely consistent with full Milne (and hence, *a fortiori*, Galilean) covariance. All that we need to assume is that the electric field, with (fibre) space components E_i, and the magnetic field, with space components B^i, should be derivable from a space-covector potential field with components A_i and a space-scalar potential field A_0 according to the usual rule

$$E_i = \nabla_i A_0 - \partial_0 A_i, \quad B^i = \epsilon^{ijk} \nabla_j A_k, \tag{3.7}$$

where ϵ^{ijk} denotes the antisymmetric Euclidean measure tenor on the fibres and ∂_0 denotes partial differentiation with respect to the base coordinate t in the fibre (i.e. space) coordinate gauge under consideration. The Milne invariance requirement for the Lorentz force per unit charge $E_i + \epsilon_{ijk} v^j B^k$ (which must be proportional to the absolute acceleration (3.4) for a particle subject to no other forces) under the effect of (3.1) leads to the transformation rule

$$A_i \to A_i, \quad A_0 \to A_0 - A_i \mathrm{d}_0 z^i, \tag{3.8}$$

for the components A_ρ ($\rho = 0, 1, 2, 3, 4$) of what we shall henceforth interpret as a 4-potential, using a four-dimensional coordinate index notation based on the identification

$$x^0 = t. \tag{3.9}$$

The rule (3.8) can be recognized as being just the ordinary transformation property for the components of a four-dimensional covector under the coordinate transformation given by (3.1), which means that the 4-potential can be regarded as a

Milne-gauge independent 1-form on space–time. In an analogous manner the 4-velocity with components u^ρ defined by

$$u^i = v^i, \quad u^0 = 1, \tag{3.10}$$

can be regarded as a Milne-gauge independent (contravariant) space–time vector, since it transforms under (3.1) according to the rule

$$u^i \to u^i + u^0 \mathrm{d}_0 z^i, \quad u^0 \to u^0. \tag{3.11}$$

This 4-vector is just the proper time derivative tangent to the space–time trajectories, as introduced in the previous section (equation (2.1)). Since the Newtonian space–time connection, and more particularly the space–time measure (which is all that is needed to define the 4-divergence) given by the product of the time-base measure and the Euclidean space measure in the fibres, are well defined independently of any choice of Milne gauge, the explicit Newtonian form of the particle number current source equations (2.2), namely

$$\partial_0 n_X + \nabla_i (n_X v^i) = r_X, \tag{3.12}$$

will be also well defined independently of gauge, the left-hand side as a whole being Milne invariant even though the separate terms are not.

4. Conversion of Newtonian hydrodynamic equations to standard form

After these kinematical preliminaries, we now come to consider the basic Eulerian dynamical equation of motion of the fluid, whereby the absolute acceleration of the particle trajectories, as defined by (3.4), which for a continuum is given by

$$\mathrm{D}_0 v^i = \partial_0 v^i + v^j \nabla_j v^i + \nabla^i \Phi, \tag{4.1}$$

is set proportional to the sum of the electromagnetic and pressure gradient forces, i.e.

$$\rho_m \mathrm{D}_0 v_i - \rho_e (E_i + \epsilon_{ijk} v^j B^k) = -\nabla_i P, \tag{4.2}$$

where each side taken as a whole is Milne invariant (even though the individual terms inside the brackets on the left-hand side are not) and the proportionality factor ρ_m is the mass density as given by

$$\rho_m = n_X m^X, \tag{4.3}$$

where the parameters m^X (representing the mass, if any, per unit particle number), like their analogues the charge parameters e^X, are chemically contravariant constants. More explicitly, we therefore have

$$n_X(\partial_0 p_i^X + v^j \nabla_j p_i^X + \nabla_i m^X \Phi) + n_X e^X (\partial_0 A_i - \nabla_i A_0 + 2 v^j \nabla_{[j} A_{i]}) = -\nabla_i P, \tag{4.4}$$

in which each separate term is chemically (but not Milne) invariant, where square brackets around indices denote the antisymmetric average over permutations, and where for each constituent we have introduced the dynamical 3-momentum per unit particle number density as defined by

$$p_i^X = m^X v_i, \tag{4.5}$$

(which will of course be zero for any constituent, such as the entropy, that is massless in Newtonian theory).

In order for the system of equations (3.12) and (4.2) to determine the time

evolution of the system, it is of course necessary to have a well-defined equation of state specifying the pressure P as a function of the densities n_X. In practice it is usually most convenient to start from a basic equation of state giving the (internal) energy density, ϵ say, in terms of the densities n_X, and then to use the chemical potentials specified by the partial variation formula

$$\delta\epsilon = \chi^X \delta n_X \tag{4.6}$$

to calculate the pressure from the formula

$$P = n_X \chi^X - \epsilon, \tag{4.7}$$

that results from relating the energy density to the work of compression of the pressure, which gives the handy variation formula

$$\delta P = n_X \delta\chi^X. \tag{4.8}$$

Since (provided the entropy is taken into account as well as the material constituents) the chemical, nuclear or other purely internal interactions among the constituents can give no net contribution to the total energy density, the rates appearing in (3.12) must satisfy the restriction

$$r_X \chi^X = 0, \tag{4.9}$$

in addition, of course, to the Newtonian mass conservation requirement,

$$\nabla_\sigma(\rho_m u^\sigma) = 0, \quad r_X m^X = 0. \tag{4.10}$$

Before continuing we remark that, just as the foregoing dynamical system is unaffected not only by the Milne gauge transformations but also by the familiar class of potential gauge transformations

$$\Phi \to \Phi + C, \tag{4.11}$$

and

$$A_0 \to A_0 + \partial_0 \psi, \quad A_i \to A_i + \nabla_i \psi, \tag{4.12}$$

where C is an arbitrary constant and ψ an arbitrary variable function over space–time, so also (in Newtonian, unlike relativistic theory) the chemical potentials are defined only modulo analogous transformations of the form

$$\chi^X \to \chi^X + C^X, \tag{4.13}$$

where the C^X are constants representing arbitrary changes in the origin of measurement of the energy per particle of the corresponding distinct constituents. The transformations (4.13) amount to a recalibration

$$\epsilon \to \epsilon + n_X C^X, \tag{4.14}$$

in the definition of the total internal energy density, but the dynamically relevant pressure as given by (4.7), and its variation as given by (4.8), will of course remain unaffected by such adjustments.

For each separate constituent, we now introduce a quantity $\mathscr{E}^X$ that is effectively its total (internal plus external) energy per particle (or more precisely energy per unit number density in cases such as the entropy for which the concept of particle may not be strictly appropriate) by the definition

$$\mathscr{E}^X = \chi^X + m^X \Phi - e^X A_0 + \tfrac{1}{2} v^i p_i^X. \tag{4.15}$$

(In the particular case of the entropy, the only term effectively present would be its chemical potential, namely the thermodynamic temperature, Θ say.) This enables us

to express the chemical potential variations appearing in the pressure variation formula (4.8) in the form

$$\delta\chi^X = \delta\mathscr{E}^X + e^X\delta A_0 - v^i\delta p_i^X - m^X\delta\Phi. \tag{4.16}$$

Using this to eliminate the pressure from the dynamic equation (4.4) gives

$$n^X(\partial_0\pi_i^X + 2v^j\partial_{[j}\pi^X{}_{i]} + \partial_i\mathscr{E}^X) = 0, \tag{4.17}$$

whose contraction with the 3-velocity components v^i reduces simply to

$$n_X v^i(\partial_0\pi_i^X + \partial_i\mathscr{E}^X) = 0, \tag{4.18}$$

where the canonical 3-momentum components are defined in terms of the ordinary 3-momentum components by a relation of the usual form

$$\pi_i^X = p_i^X + e^X A_i, \tag{4.19}$$

and where the antisymmetrization has permitted us to use partial coordinate derivatives, as indicated by ∂_i, in place of the covariant derivatives, that were indicated by ∇_i.

To complete the specification of the components π_ρ^X of the required 4-momentum covector, we now define its time component simply by

$$\pi_0^X = -\mathscr{E}^X. \tag{4.20}$$

It then follows that the contraction λ^X (as introduced in (2.6) by the definition (2.7)) of the 4-momentum with the 4-velocity will be related to the effective energy per particle $\mathscr{E}^X$ by a formula of the ordinary Legendre type for the relation between an ordinary Lagrangian and the corresponding Hamiltonian energy function, i.e.

$$\lambda^X = v^i\pi_i^X - \mathscr{E}^X. \tag{4.21}$$

It has already been observed that the 4-velocity vector and the electromagnetic 4-potential 1-form are Milne invariant in the sense that the combined effect of the Milne transformation (3.1) (leading to (3.11) and (3.8) respectively) and an ordinary four-dimensional coordinate change back to the original reference system leaves them unchanged:

$$w^\rho \to w^\rho, \tag{4.22}$$

and

$$A_\rho \to A_\rho. \tag{4.23}$$

However the 4-momentum 1-forms (as defined by (4.19) and (4.20)) have a non-trivial Milne gauge dependence, since the effect of such a Milne-plus-coordinate transformation is to induce additive adjustments expressible (in flat space coordinates) as

$$\pi_\rho^X \to \pi_\rho^X + \partial_\rho(m^X x^i \mathrm{d}_0 z_i) + \tfrac{1}{2}m^X(\mathrm{d}_0 z^i)(\mathrm{d}_0 z_i)\partial_\rho t. \tag{4.24}$$

Although the 4-momenta are thus affected by general Milne transformations and even by Galilean transformations (i.e. those for which $\mathrm{d}_0 z^i$ is itself independent of t), as well as being effected by the gauge transformations (4.11), (4.12), and (4.13) of the gravitational, electromagnetic and chemical potentials, it is to be noticed that the relevant additive correction terms are all exact (i.e. pure gradient) 1-forms which means that the corresponding generalized vorticity 2-forms defined by their exterior derivatives will be invariant, i.e.

$$\partial_{[\rho}\pi^X{}_{\sigma]} \to \partial_{[\rho}\pi^X{}_{\sigma]} \tag{4.25}$$

with respect to all these kinds of (Milne and other) gauge transformations.

The standard form (2.5) of the equations of motion is equivalent to the statement that the sum of the contractions of these vorticity 2-forms with the current 4-vectors of the corresponding constituents should vanish, i.e.

$$n^{\rho}_{X}\,\partial_{[\rho}\pi^{X}{}_{\sigma]}=0. \tag{4.26}$$

To verify this basic result, all that remains is to observe (bearing in mind the definition (3.10) of the 4-velocity components w^{ρ}) that (4.17) and (4.18) respectively are precisely the space and the time components of the original version (2.5) of the required result.

To complete our demonstration, for the Newtonian case, of the formulae presented in §2, it suffices to remark that the result of substituting (4.16) in the pressure variation formula (4.8) can be expressed in four-dimensional notation as

$$\delta P=-n^{\rho}_{X}\,\delta\pi^{X}_{\rho}+j^{\rho}\delta A_{\rho}-\rho\delta\Phi, \tag{4.27}$$

which reduces directly to the previously stated expression (2.3) if the gravitational field Φ is held fixed. (It is only when the final gravitational adjustment term is present that this Newtonian form differs from the corresponding general relativistic pressure variation formula as given in the Appendix.)

5. Formulation in terms of a chemically preferred reference current

Having shown how the standard form (2.5) of the fluid equation of motion and the corresponding general form (2.3) of the pressure variation formula can be derived in a special case (i.e. the Newtonian limit) we shall now proceed immediately to consider some of the direct consequences that can be drawn quite generally just from these two equations and the associated divergence laws (2.2) together with the elementary definitions (2.1) and (2.4). (The derivation of these equations in the technically simpler and physically less specialized context of special and general relativistic theory will be postponed until the Appendix.)

Unlike the work of the preceding sections (and the Appendix) the results of §§6, 7, and 8 will be valid only for the case of a fluid that is perfect in the strict sense that all the dynamically relevant constituents are conserved, i.e. the corresponding creation rates r_{X}, as they appear in the divergence conditions (2.2), are all zero, so that the system of equations of motion is time-reversible.

Before imposing this condition however, we remark more generally that even if only some, but not all linear combinations of the constituent number of currents is conserved, it will often be convenient to choose one of these conserved currents as a basic reference for accounting purposes. If the chosen reference number density, n say, is given by

$$n=N^{X}n_{X} \tag{5.1}$$

for some set of constant coefficients N^{X}, then the desired conservation property,

$$\nabla_{\rho}n^{\rho}=0, \tag{5.2}$$

as expressed using the obvious notation

$$n^{\rho}=nw^{\rho}, \tag{5.3}$$

implies the requirement that the creation rates should satisfy the restriction

$$N^{X}r_{X}=0. \tag{5.4}$$

In many physical contexts an appropriate choice for such a reference density will be the baryon number density. (In the example presented at the end of §2, the baryon number density would be obtainable from the original molecular frame by taking $N^1 = 18$, $N^2 = 32$, which is equivalent to taking the atomic frame components $N'^1 = 1$, $N'^2 = 16$.)

In so far as relativistic theory is concerned there is no loss of generality in choosing to use any such set of coefficients N^X for the purpose of associating corresponding mass constants m^X with the constituents by a proportionality relation of the form

$$m^X = mN^X, \tag{5.5}$$

where the proportionality factor m is a constant interpretable as a mean mass per reference particle, so that the corresponding 'rest' mass density will be given by

$$\rho_m = mn. \tag{5.6}$$

It will also be always possible, and for many purposes most convenient, to impose the requirements (5.5) and (5.6) in the case of Newtonian theory, but in this case it would effectively entail a restriction on the choice of the N^X, since (unlike the more generally applicable relativistic theory, for which the constants m^X may be chosen arbitrarily as far as the general mathematical formalism presented in the Appendix is concerned) the structure of the (physically less accurately realistic) Newtonian theory is such that the constituent particle masses and the corresponding mass density (as originally defined by (4.3)) will be unambiguously determined at the outset.

With respect to any such reference number density n (regardless of whether, in the Newtonian case, the restriction (5.5) has been made) the composition of the fluid will be describable in terms of the set of scalar constituent number density ratios

$$\nu_X = \frac{n_X}{n}, \tag{5.7}$$

for whose time evolution the divergence conditions (2.2) will give the simple formulae

$$u^\rho \partial_\rho \nu_X = \frac{r_X}{n}, \tag{5.8}$$

which (by (5.3)) will obviously preserve the necessary normalization condition

$$N^X \nu_X = 1, \tag{5.9}$$

restricting the independence of the ratios ν_X. These ratios can be used to introduce what is interpretable as a mean 4-momentum or equivalently as the total 4-momentum per unit reference particle number, by the obvious definition

$$\pi_\rho = \nu_X \pi_\rho^X. \tag{5.10}$$

In terms of such a mean 4-momentum 1-form, the generalized Bernoulli theorem (2.13) reduces to the form of a scalar conservation law,

$$u^\rho \partial_\rho (k^\sigma \pi_\sigma) = 0, \tag{5.11}$$

where the conserved scalar $k^\sigma \pi_\sigma$ is interpretable, depending on the geometric nature of the symmetry generator k^ρ, as representing the mean energy, angular momentum, or whatever it may be, per particle.

6. Generalized Clebsch formulation

In this and the remaining §§7 and 8 we shall suppose that all the creation rates are zero

$$r_X = 0, \tag{6.1}$$

which means that the choice of the constituent reference vector with components N^X specifying the choice of a current satisfying (5.4) may be taken arbitrarily, so that its introduction as a formally preferred reference vector in the constituent vector space will not entail any real loss of chemical covariance. Under these circumstances we can rewrite the basic dynamic equation of motion (2.5) in the simple alternative form

$$2u^\rho \partial_{[\rho} \pi_{\sigma]} = -\lambda^X \partial_\rho \nu_X, \tag{6.2}$$

where the coefficients λ^X of the number ratio gradients on the right-hand side are the momentum-velocity contractions given by (2.7), whose interpretation as constituent particle Lagrangians is motivated by (4.21).

Subject to the imposition of (5.5) (which, as emphasized above, is merely a definition in the relativistic case, although a real restriction on the chemical covariance of the choice of n in the Newtonian case) one may regroup the terms appearing on the right-hand side of (6.2) so as to obtain the alternative expression

$$2u^\rho \partial_{[\rho} \pi_{\sigma]} = \frac{1}{n}\partial_\sigma \epsilon - (\epsilon + P)\,\partial_\sigma \frac{1}{n} - u^\rho A_\rho \partial_\sigma \frac{\rho_e}{n}. \tag{6.3}$$

wherein only the first two terms on the right-hand side will remain when electromagnetic effects are absent.

The modified strictly conservative version (6.3) of the dynamic equation of motion of the fluid can be used as the starting point of a generalized Clebsch-type formulation that includes as restricted special cases both the Newtonian formulation of Seliger & Whitham (1968) and the relativistic formulation of Schmidt (1970) and Schutz (1970). The first step is to introduce a (chemically contravariant) set of potentials ϕ^X that may be interpreted as Jacobian action variables along the world lines for the corresponding kinds of particular constituent according to the prescription

$$u^\rho \partial_\rho \phi^X = \lambda^X. \tag{6.4}$$

(These action variables generalize the quantity originally introduced by van Danzig (1939) under the name of 'thermasy' for a special case in which there was only one constituent – namely the entropy – that was relevant in the sense of its density being independent of that of the reference particles.)

In terms of these fields ϕ^X (which introduce a new kind of gauge dependence in so much as their precise specification is dependent on the choice of a set of independently arbitrary hypersurfaces as starting positions for the measurement of the corresponding actions along the flow lines) one can define a modification of the 4-momentum 1-form, with components given by

$$\kappa_\rho = \pi_\rho - \nu_X \partial_\rho \phi^X, \tag{6.5}$$

which is such that the flow 4-vector has vanishing contraction not only with the 1-form itself, i.e.

$$u^\rho \kappa_\rho = 0, \tag{6.6}$$

but also with its exterior derivative, i.e.

$$u^\rho \partial_{[\rho} \kappa_{\sigma]} = 0. \tag{6.7}$$

As was emphasized in the previously cited discussion of Carter (1979), these are the necessary and sufficient conditions for the form to be strongly conserved by the flow in the sense of having vanishing Lie derivative with respect to any flow tangent vector field, σw^ρ (where the time rescaling factor σ is an arbitrarily variable scalar field over space–time), i.e.

$$[\sigma \boldsymbol{u} \mathscr{L}] \kappa_\rho = 0. \tag{6.8}$$

This means that in any coordinate system comoving with the fluid (no matter how the time parametrization is adjusted) the components κ_ρ of this 1-form will have time-independent values.

It follows from any two of (6.6), (6.7), (6.8) that this 1-form is the inverse projection of a corresponding 1-form on the three-dimensional quotient manifold of space–time by the congruence of flow trajectories. It follows that as in the historical three-dimensional case considered by Clebsch, one can apply the theorem of Pfaff (see e.g. Schutz 1970) to the effect that this three-dimensional 1-form must be (locally) the sum of the gradient of a potential γ, say, and of another term proportional to the gradient of another potential designated as β. This in turn implies that κ_ρ can be expressed as a sum of the same form in four dimensions, where γ, β, and the proportionality factor, α say, are all constant along the flow lines. In this particular application of Pfaff's theorem the first potential, γ, can be seen to be redundant, because without loss of generality it can be cancelled out by using the freedom to make gauge transformations

$$\phi^X \to \phi^X - \gamma^X, \tag{6.9}$$

while still preserving the defining conditions (6.4), where the γ^X are any scalar fields satisfying the condition,

$$w^\rho \partial_\rho \gamma^X = 0, \tag{6.10}$$

of constancy along the flow lines. We thus conclude that, while retaining the freedom to make independent transformations of the form (6.9) to all but one of the action potentials ϕ^X, one will be able to adjust the last one in such a way that the conserved 1-form κ_ρ will be expressible in the form

$$\kappa_\rho = \alpha \partial_\rho \beta, \tag{6.11}$$

with

$$w^\rho \partial_\rho \alpha = 0, \quad w^\rho \partial_\rho \beta = 0. \tag{6.12}$$

which have the same form as the number ratio conservation laws

$$w^\rho \partial_\rho \nu_X = 0, \tag{6.13}$$

obtained from (5.8) when (6.1) holds. Returning via (6.5) to the original momentum 1-form, (6.11) gives a Clebsch-type expression of the very simple but quite generally valid form

$$\pi_\rho = \alpha \partial_\rho \beta + \nu_X \partial_\rho \phi^X. \tag{6.14}$$

Working backwards, one can verify straightforwardly step by step that in conjunction with (6.13) the original dynamic equation of motion (6.2) can conversely be recovered from, and is therefore equivalent to, the evolution equations (6.4) and (6.12) together with the relation (6.14).

The required generalized Clebsch formulation therefore consists of just the four equations (6.4), (6.12), (6.13), (6.14), together with the conservation law (5.2) for the reference current (5.3) which one needs in any case (bearing in mind that only three of the four components of the dynamic equation (6.2) are algebraically independent), supplemented of course by the purely algebraic equations (equation of state and definition of momentum in terms of velocity) that relate the current to the 4-

momentum 1-form (but which cannot be made more explicit at this point, since any more detailed specification would necessitate distinguishing Newtonian from relativistic theory, contrary to the express purpose of the present section).

The potential gradient expression (6.14) for the 4-momentum can be rewritten in a form that more directly generalizes the traditional version of the Clebsch formulation (Schutz 1970) by introducing the mean action potential per reference particle, as defined by

$$\phi = \nu_X \phi^X \tag{6.15}$$

which can be seen from (6.4) and (6.13) to satisfy

$$u^\rho \partial_\rho \phi = \lambda, \tag{6.16}$$

where

$$\lambda = \nu_X \lambda^X \tag{6.17}$$

defines what is interpretable as a mean Lagrangian in view of the alternative expression

$$\lambda = u^\rho \pi_\rho. \tag{6.18}$$

Substitution of (6.15) in (6.14) gives the alternative expression

$$\pi_\rho = \partial_\rho \phi - \phi^X \partial_\rho \nu_X + \alpha \partial_\rho \beta. \tag{6.19}$$

When working with this traditional version (in which ϕ is the historic descendant of the potential originally introduced by Clebsch himself) one does not need to postulate an independent additional evolution equation for ϕ because (6.16) can be seen from (6.18) to be a direct consequence of (6.12), (6.13) and of the new relation (6.19) which replaces the basic equation (6.18) of our first version of the Clebsch formulation as summarized in the preceding paragraph.

7. Standard version of potential formulation and pressure variation principle

Although the customary approaches to any Clebsch-type potential gradient formulation (and in particular the very definition of the mean potential ϕ appearing in the expression (6.19) that generalizes the traditional Clebsch-type formulae) depend on the use of a particular chosen number density of the form (5.1) as a reference, there is no difficulty in obtaining a closely related potential gradient formulation that is chemically covariant in the strong sense, meaning that no particular chemical reference vector is involved.

One way of obtaining this chemically covariant version, which we shall refer to as the standard potential formulation (since it bears the same relationship to our standard dynamic equation (4.25) as does the preceding version (6.19) to the reference based dynamic equation (6.2)) is simply to multiply out the reference number density n from our earlier Clebsch-type expression (6.14). What we obtain is a formula for the total 4-momentum density 1-form, Π_ρ, as defined by (2.14), which is manifestly a chemical invariant. The resulting standard potential formula may be expressed concisely as

$$\Pi_\rho = n_\Sigma \partial_\rho \phi^\Sigma, \tag{7.1}$$

where (as in the case of small case space indices) the change from Latin to Greek capital indices denotes that an extra dimension has been added to the vector space of chemical constituents, so that for example if we take X ranging from 0 to N say then we can take Σ ranging from -1 to N. The extra number density, which in this case will be denoted by n_{-1} may be accounted for as the product $n\alpha$ of the scalar fields

n and α introduced in the preceding section, but one does not need to know this antecedence in order to proceed: all that matters is that it should be conserved like the other (physical) number densities so that for all values of Σ we have

$$\nabla_\rho(n_\Sigma u^\rho) = 0. \tag{7.2}$$

We may similarly extend the set of equations for the evolution of the action potentials to

$$u^\rho \partial_\rho \phi^\Sigma = \lambda^\Sigma, \tag{7.3}$$

where, since the extra potential has as its antecedent the β of the proceeding section, we must set the corresponding 'extra Lagrangian' equal to zero, i.e.

$$\lambda^{-1} = 0 \tag{7.4}$$

with the indexation system proposed above. (As a specific example, if we were considering a Newtonian system in which the index 0 was chosen to represent the entropy, while the other values were taken to indicate N different kinds of material particle constituents, then the zeroth Lagrangian field would just be the negative of the temperature, i.e. $\lambda^0 = -\Theta$, while the others would have the general form, $\lambda^X = \frac{1}{2}m^X v^i v_i - \chi^X + e^X(A_0 + v^i A_i) - m^X \Phi$. In the relativistic case the general form will have the simpler expression given at the end of the Appendix.)

In this standard formulation the fluid system is fully described by the evolution equations (7.2) and (7.3) and by our potential formula (7.1) (considered as defining the total momentum density as a function of the potential gradients) together with the purely algebraic equations (7.4) and (2.7) (for the source terms in (7.3)) subject of course to specification of the equation of state and to the specification of the relation between 4-momentum and velocity, which depends on whether the system is Newtonian or relativistic.

One of the traditional motivations for setting up a Clebsch-type formulation is in order to be able to obtain an ordinary fixed space–time position (Eulerian) field-variation principle for the differential equations of the system, which, as has been shown by Schutz & Sorkin (1977), is not possible if one restricts oneself to the ordinary physical (in physics the word 'physical' usually means independent of some pertinent gauge dependence) variables, n^ρ_X without allowing oneself the use of the extra current, n^ρ_{-1} and the potentials ϕ^Σ. Such a procedure can be carried through in the very general (indifferently Newtonian or relativistic) context under consideration here by using the fact that the dependence of the pressure, on which such variation principles are based, on the variations of the 4-momenta can always be expressed in the form

$$\delta P = -n_X u^\rho \delta \pi^X_\rho + j^\rho \delta A_\rho + \delta_G P, \tag{7.5}$$

where the last term, $\delta_G P$ is zero if the gravitational field is held fixed, which will be supposed to be the case as far as the present section is concerned. This last term is the only one whose explicit form would depend on whether one is using Newtonian theory (in which case it can be read out from (4.27)) or relativistic theory (in which case it can be read out from the analogous formula (A 33) in the Appendix). For the purpose of the present section we shall also be able to ignore the second term, since we are not concerned with the source equations for the electromagnetic field whose variation may therefore be set equal to zero.

When the last two external field variation terms vanish, i.e.

$$\delta A_\rho = 0, \quad \delta_G P = 0, \tag{7.6}$$

one is left with a pressure variation contribution which can be rewritten in terms of the chemically invariant total 4-momentum density as

$$\delta P = \lambda^X \delta n_X - u^\rho \delta \Pi_\rho. \tag{7.7}$$

If we now consider the potentials and densities of the potential formulation set up above as the independent variables, then we may use (7.1) directly to evaluate the first term in (7.7) so as to obtain

$$\delta p = (\lambda^\Sigma - u^\rho \partial_\rho \phi^\Sigma)\, \delta n_\Sigma + \nabla_\rho (n_\Sigma u^\rho)\, \delta \phi^\Sigma - \nabla_\rho (n_\Sigma u^\rho \delta \phi^\Sigma). \tag{7.8}$$

When this is substituted into a local variation of the space–time 4-volume integral of the pressure

$$I = \int p \, \mathrm{d}^{(4)} V, \tag{7.9}$$

then the divergence term goes out as usual by Green's theorem, leaving

$$\delta I = \int \{(\lambda^\Sigma - u^\rho \partial_\rho \phi^\Sigma)\, \delta n_\Sigma + \nabla_\rho (n_\Sigma u^\rho)\, \delta \phi^\Sigma\} \, \mathrm{d}^{(4)} V. \tag{7.10}$$

The requirement that this should vanish for arbitrary variations of the n_Σ (including the unphysical one, n_{-1}), and of all the ϕ^Σ, leads immediately to the basic evolution equations (7.2) and (7.3) of the system.

(It is a straightforward exercise to translate this general pressure variation principle back into the more traditional reference density based form, in which the quantities to be varied are taken to be the ratios ν_X and the potentials α, β, of the previous section, together with the same action potential ϕ^X as in the present standard version (7.10). Since the ν_X are restricted by (5.9) there will be one less independent equation for the evolution of the ϕ^X, but the equation that is apparently lost this way is in any case not independent, being recoverable in the form (6.16) as an identity resulting automatically from (6.12) and (6.14).)

8. Derivation of the canonical form from the standard form

In order to be able to convert the equations of motion from the general standard form obtained in the preceding work to the more specialized canonical form whose particularly convenient implications have been described elsewhere (Carter 1979) we must restrict ourselves to the situation in which not more than two of the constituents are independent, so that all the internal functions will be expressible in terms of just two number densities (which in typical applications might correspond to baryons and entropy or to ions and electrons), let us say $n_{[0]}$ and $n_{[1]}$. Thus the number density n_X for the Xth general constituent will be determined by an expression of the form

$$n_X = N_X^{[0]} n_{[0]} + N_X^{[1]} n_X, \tag{8.1}$$

for some set of fixed coefficients $N_X^{[0]}$, $N_X^{[1]}$. There will be no further loss of generality in choosing $n_{[0]}$ to be the reference number density n as introduced in §5, which amounts to setting the first of the corresponding constituent number density ratios

$$\nu_{[0]} = \frac{n_{[0]}}{n}, \quad \nu_{[1]} = \frac{n_{[1]}}{n}, \tag{8.2}$$

equal to unity, i.e.

$$\nu_{[0]} = 1. \tag{8.3}$$

This means that for the Xth general constituent the corresponding number density ratio will be given in terms of a single independent variable, namely $\nu_{[1]}$, by the inhomogeneous linear expression

$$\nu_X = N_X^{[0]} + N_X^{[1]} \nu_{[1]}, \tag{8.4}$$

so that its gradient will be given simply by

$$\partial_\mu \nu_X = N_X^{[1]} \partial_\mu \nu_{[1]}. \tag{8.5}$$

Hence in terms of the correspondingly transformed values

$$\lambda^{[0]} = \lambda^X N_X^{[0]}, \quad \lambda^{[1]} = \lambda^X N_X^{[1]}, \tag{8.6}$$

of the constituent Lagrangian scalar fields as defined by (2.7), we can carry out the sum-over-constituents in the chemically covariant expression (2.6) of the equation of motion so as to obtain the (reference combination dependent) form

$$[u\mathscr{L}]\pi_\mu = \partial_\mu \lambda - \lambda^{[1]} \partial_\mu \nu_{[1]} \tag{8.7}$$

where π_μ and λ are respectively the total 4-momentum and 'Lagrangian function' per reference particle, as defined by (5.10) and (6.17) or (6.18). The analogous evaluation of the sum-over-constituents in the standard form (2.5) itself leads to the even simpler alternative expression

$$2u^\rho \partial_{[\rho} \pi_{\mu]} = -\lambda^{[1]} \partial_\mu \nu_{[1]}. \tag{8.8}$$

To obtain the desired canonical form of the equations of motion, it now suffices to rescale the flow 4-vector suitably, replacing u^μ by

$$V^\mu = V u^\mu, \tag{8.9}$$

where the (locally variable) time dilatation factor V is chosen to be given by

$$V = -1/\lambda^{[1]}. \tag{8.10}$$

(In the particularly simple case for which the independent constituents are just baryon number and entropy, V would thus be just the inverse of the corresponding local temperature, Θ.)

In terms of the correspondingly rescaled reference particle Lagrangian scalar field

$$L = V\lambda^{[0]}, \tag{8.11}$$

and the associated Hamiltonian scalar field

$$H = -\nu_{[1]}, \tag{8.12}$$

as defined in accordance with the Legendre relation

$$L + H = V^\mu \pi_\mu, \tag{8.13}$$

the system of equations of motion (8.7) is convertible into Lagrangian form (Carter 1979) as

$$[V\mathscr{L}]\pi_\mu = \partial_\mu L, \tag{8.14}$$

while the required canonical form (i.e. the corresponding Hamiltonian version of the system) is obtained from (8.8) as

$$2V^\rho \partial_{[\rho} \pi_{\mu]} = -\partial_\mu H. \tag{8.15}$$

The specially simple uniformly canonical form, as characterized by the further restriction that the term on the right-hand side should be absent from the basic general canonical form (8.15), can be seen to occur in the barytropic case in which not two but only a single one of the constituents is effectively independent so that $\nu_{[1]}$ has a uniform value throughout the fluid.

Appendix. Derivation of the standard form in the relativistic case

So long as one is not concerned with the gravitational source equations it is actually simpler to formulate the theory for the (special or general) relativistic case in which the kinematic space–time background is described straightforwardly by an ordinary (non-degenerate, and therefore usable for raising and lowering of indices) pseudo-Riemannian metric,

$$ds^2 = g_{\rho\sigma}\,dx^\rho\,dx^\sigma, \tag{A 1}$$

than for the Newtonian case whose description requires the use of the somewhat subtler concept of the Milne bundle structure, as introduced in §3 (or of some other equivalent, and no less elaborate, alternative mathematical description) in which the metric (A 1) has degenerated into a purely contravariant (usable only for raising but not lowering of indices) symmetric tensor components $\gamma^{\rho\sigma}$ having as a null eigenvector the separate time-gradient 1-form with components $\partial_\rho t$. Natural analogues of these degenerate Newtonian residual parts of the space–time metric are determined in the presence of a fluid by the corresponding local rest frame. The (no longer closed) analogue of the (closed) Newtonian time metric form given (in the notation of §3) by $\partial_\rho t$ will have components obtainable in the form $c^{-2}u_\rho$ from the proper 4-velocity, with components u^ρ, via an ordinary index lowering operation,

$$u_\rho = g_{\rho\sigma}\frac{dx^\sigma}{d\tau}, \quad d\tau^2 = -c^{-2}\,ds^2, \tag{A 2}$$

(in which we have explicitly included a dimensional parameter, c, representing the speed of light, which may be considered to tend to ∞ in the degenerate Newtonian limit, but which for other purposes may be set equal to one by choice of units). Like their Newtonian analogue, $c^2\partial_\rho t$, the components u_ρ will satisfy a null eigenvalue equation of the form

$$\gamma^{\rho\sigma}u_\sigma = 0, \tag{A 3}$$

where, in the relativistic case, the degenerate (positive indefinite) tensor with components $\gamma^{\rho\sigma}$ is defined by the metric decomposition

$$g^{\rho\sigma} = \gamma^{\rho\sigma} - c^{-2}u^\rho u^\sigma, \tag{A 4}$$

which of course embodies the proper-time normalization condition,

$$u_\rho u^\rho = -c^2, \tag{A 5}$$

resulting from (A 2).

In terms of the foregoing decomposition the relativistic perfect fluid is postulated to have an energy-momentum tensor given directly by

$$T^{\rho\sigma} = \rho u^\rho u^\sigma + P\gamma^{\rho\sigma}, \tag{A 6}$$

where P is the pressure, as before, and where ρ is interpretable as the local mass-energy density in the fluid rest frame. When the only non-gravitational external force is electromagnetic, the dynamic equations of motion take the form

$$\nabla_\sigma T^\sigma_\rho = F_{\rho\sigma}j^\sigma, \tag{A 8}$$

where the electromagnetic field tensor is the exterior derivative of the 4-potential 1-form, i.e.

$$F_{\rho\sigma} = 2\,\partial_{[\rho} A_{\sigma]}, \tag{A 9}$$

the gravitational forces (if any) being taken into account automatically by taking the covariant differentiation operation indicated by V to be defined in terms of the (possibly curved) Riemannian connection associated with the metric (A 1).

The full system of differential equations of motion of the fluid consists of the pseudo-conservation law (A 8) (it will only be a strict conservation law in the special relativistic flat space–time case, i.e. when no gravitational effects are taken into account) together with the (true) conservation or creation laws (2.2) for the independent conserved constituents (including the entropy if thermal effects are relevant). To complete the specification of the system, these differential equations must of course be supplemented by the necessary algebraic relations, namely the equation of state, which may be thought of as specifying the mass-energy density ρ as a function of the independent (proper) constituent number densities n_X, and finally the expressions for the creation rates r_X appearing in (2.2) as a result of any (chemical, nuclear, or other) reactions that may be relevant. Except for the index value $X = 0$ say, corresponding to the entropy, the rates r_X may be expressed in terms of the rates $r_{[C]}$ say (per unit proper volume) of the various particular reactions that may be going on (where $[C]$ is an index labelling the distinct reactions involved) by an expression of the form

$$r_X = N_X^{[C]}\, r_{[C]}, \tag{A 10}$$

where the coefficients $N_X^{[C]}$ are integer-valued constants representing the number of particles of the corresponding particle created (or, for negative values, destroyed) in the reaction concerned. The reaction rates $r_{[C]}$ are supposed to be specified in terms of the constituent densities, including the entropy, in such a way as to ensure that the latter has a non-negative creation rate, $r_0 \geqslant 0$, as determined, not by a direct relation of the form (A 10) but by the consistency condition to be given below.

Since consistency with the normalization condition (A 5) requires that only three of the four components of the dynamic equation (A 8) can be independent, the functional dependence of the pressure, P, is determined in terms of that for ρ by the requirement the intrinsic energy conservation law

$$u^\rho \nabla_\sigma T_\rho^\sigma = 0, \tag{A 11}$$

(obtained as the projection of (A 8) along the flow, taking account of the restriction (2.4) expressing the absence of electric conductivity) should be satisfied as an identity. The explicit form of this required identity is obtained by substitution of (A 6) as

$$(\rho + c^{-2}P)\,\nabla_\sigma u^\sigma + u^\sigma \nabla_\sigma \rho = 0. \tag{A 12}$$

Introducing what may be described as the relativistic chemical potentials μ^X (which we shall see to be alternatively interpretable as effective particle mass parameters) by the partial variation formula

$$\delta\rho = \mu^X\, \delta n_X, \tag{A 13}$$

and using the separate constituent creation or conservation laws (2.2) one sees that (A 12) may be rewritten as

$$(\rho + c^{-2}P - \mu^X n_X)\,\nabla_\sigma u^\sigma = \mu^X r_X. \tag{A 14}$$

In order for this to hold as an identity, the net chemical potential energy release

(including the otherwise unspecified entropy contribution, $\mu^0 r_0$, where μ_0 is the thermodynamic temperature Θ) must vanish separately,

$$\mu^X r_X = 0, \tag{A 15}$$

and the equation of state for P must be specified in terms of that for ρ by

$$P = (n_X \mu^X - \rho) c^2, \tag{A 16}$$

for which the corresponding partial variation formula is

$$\delta P = c^2 n_X \delta \mu^X. \tag{A 17}$$

For the purpose of comparison with analogous Newtonian formulae, it is a very common practice even in a relativistic context to introduce the concept of a constant ('rest') mass per particle, m^X say, for each constituent. In terms of such constants, one can if one wishes define the relativistic generalization of the Newtonian chemical potentials χ^X (as introduced by (4.6)) by the decomposition

$$\mu^X = m^X + c^{-2} \chi^X, \tag{A 18}$$

thereby ensuring that (4.8) will still be obeyed even in the relativistic case. The corresponding generalization of the energy density ϵ, constructed so as to preserve (4.6) and (4.7) also in the relativistic case must then be taken to be given by the decomposition

$$\rho = n_X m^X + c^{-2} \epsilon. \tag{A 19}$$

As far as the intrinsic mathematical structure of the theory is concerned however, such a decomposition is an unnecessary complication, the choice of the mass constants m^X being quite arbitrary, since any redefinition by adjustments of the form

$$m^X \to m^X - c^{-2} C^X, \tag{A 20}$$

(where the C^X are freely chosen constants) may be allowed for by corresponding chemical potential gauge adjustments of the form that has already been specified by (4.13) and (4.14). Such recalibrations of the χ^X and of ϵ have no effect on the relativistically well-defined (gauge independent) quantities μ^X and ρ as introduced by (A 13) and (A 6). The arbitrariness may of course be somewhat restricted if, as will usually be the case, one also desires to recover the Newtonian chemical-energy-balance condition (4.9) from its relativistic analogue (A 15): for this purpose the quantities m^X will have to satisfy restrictions of the same form as the Newtonian mass conservation law, i.e.

$$m^X r_X = 0. \tag{A 21}$$

Subject to the convention that a zero (rest) mass value be attributed to the entropy,

$$m^0 = 0, \tag{A 22}$$

the restriction (4.9) can be replaced by the more explicit requirements

$$m^X N_X^{[C]} = 0 \tag{A 23}$$

which will then be well defined despite the fact that the entropy coefficients $N_0^{[C]}$ are in general indeterminate. The restrictions (A 21) will guarantee a resulting conservation relation that is formally the same as the Newtonian mass conservation law (4.10), or in other words

$$\nabla_\sigma (m^X n_X u^\sigma) = 0. \tag{A 24}$$

(The restrictions (A 21) are thus effectively equivalent to the condition (5.5) for admissibility of the m^X as a possible selection for the reference-number-per-particle

coefficients N^X specifying a particular choice, $m^X n_X$, for the reference number density n.) The preservation of the restrictions (A 21) evidently entails the corresponding limitation

$$C^X r_X = 0 \tag{A 25}$$

on the freedom of choice of the constants C^X appearing in (A 19).

(From a physical point of view, although the meaning of 'rest mass' may be quite unambiguous for an isolated individual particle, the notion of the 'rest mass per particle' in a fluid is essentially subjective. It is usually supposed to mean the mass that would remain if all 'available' energy were removed. The physical ambiguity concerns the different kinds of physical process, e.g. atomic, nuclear, or GUT reactions, that might be taken into account in a particular physical context in deciding what is to be considered as available in practice. Thus to give a precise meaning to the commonly occurring expression 'rest mass per baryon', legitimate alternative choices would be the mass of a neutral hydrogen atom, or, allowing for ionization processes, the mass of a proton, or, allowing for nuclear reactions, $\frac{1}{56}$ of the mass of an ordinary iron nucleus. In a general relativistic context in which black hole formation by gravitational collapse may be envisaged, all the energy may in principle be considered to be available, so that there is a respectable physical justification for what is in any case the most convenient choice for many general mathematical purposes, namely to set all the mass parameters m^X simply to zero, and then to forget about them.)

The important role that was played by the constant mass parameters m^X in Newtonian theory, is taken over instead in relativistic theory by the variable relativistic chemical potentials, μ^X, whose significance as effective masses has long been recognized by many authors (see e.g. Thorne 1967 or Misner *et al.* 1973). What we wish to emphasize more particularly in the present approach is the importance of the corresponding dynamical 4-momentum 1-forms, as defined by

$$p^X_\rho = \mu^X u_\rho. \tag{A 26}$$

Variation of the resulting expression, in terms of the 4-momenta, for the relativistic chemical potentials,

$$\mu^X c^2 = -u^\rho p^X_\rho, \tag{A 27}$$

leads (cf. the variation formulae in Schutz & Sorkin 1977) to the relativistic analogue of the expression (4.16) giving the variation of these potentials in terms of the variations of the momenta, and also of conceivable gravitational variations of the external space–time background, in the form

$$c^2 \delta\mu^X = -u^\rho \delta p^X_\rho + \tfrac{1}{2}\mu^X u^\rho u^\sigma \delta g_{\rho\sigma}. \tag{A 28}$$

In order to obtain the required standard form of the equations of motion we now introduce the appropriate canonical 4-momenta (differing from the dynamic 4-momenta only when an electromagnetic field is present) by a relation of the usual form:

$$\pi^X_\rho = p^X_\rho + e^X A_\rho. \tag{A 29}$$

which implies that the corresponding Lagrangian scalars (as specified by (2.7)) will be given by

$$\lambda^X = -\mu^X c^{-2} + e^X u^\rho A_\rho. \tag{A 30}$$

It is then straightforward to verify that the independent equations remaining from (A 8) (after (A 16) has been used to ensure that (A 12) is satisfied as an identity), namely the orthogonally projected part consisting of the acceleration equations

$$(\rho + c^{-2}P)\, u^\sigma \nabla_\sigma u_\rho + \gamma^\sigma_\rho \nabla_\sigma P = 2\nabla_{[\rho} A_{\sigma]} j^\sigma, \tag{A 31}$$

can be rewritten in the desired form (2.5), i.e. we obtain as a final result the simple standard expression

$$n^{\sigma}_{X}\nabla_{[\sigma}\pi^{X}{}_{\rho]}=0. \tag{A 32}$$

To obtain the relativistic analogue of the pressure variation formula (4.27), all we need to do is to substitute (A 28) in (A 17), which leads to a result expressible as

$$\delta P=-n^{\rho}_{X}\,\delta\pi^{X}_{\rho}+j^{\rho}\delta A_{\rho}+\tfrac{1}{2}n^{\rho}_{X}\,\pi^{X\sigma}\delta g_{\rho\sigma}, \tag{A 33}$$

which (as in the Newtonian case) reduces to the common form (2.3) on which the standard variation formula (7.10) is based, when the gravitational space–time background is held fixed.

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